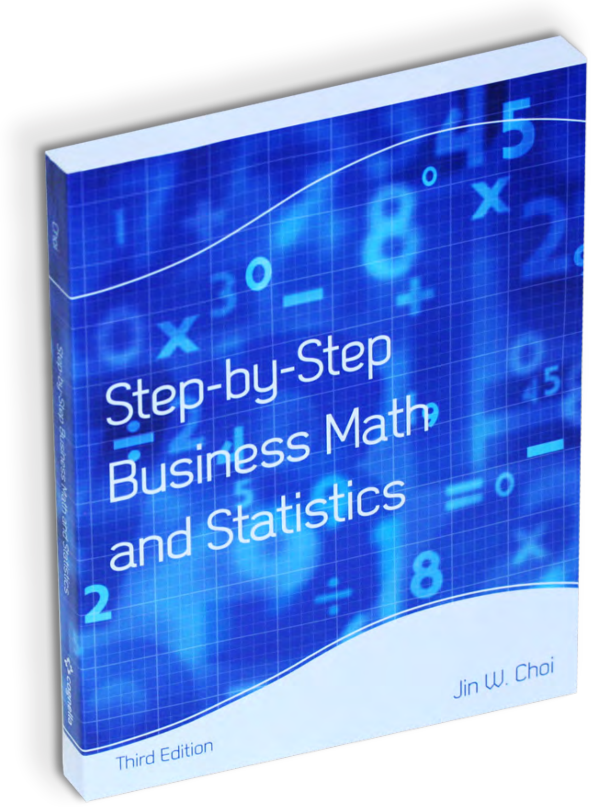
cognella academic publishing

Sneak Preview



Step-by-Step Business Math and Statistics

By Jin W. Choi

Included in this preview:

- Copyright Page
- Table of Contents
- Excerpt of Chapter 1

For additional information on adopting this book for your class, please contact us at 800.200.3908 x501 or via e-mail at info@cognella.com

3970 Sorrento Valley Blvd. | Suite 500 | San Diego, CA 92121 | 800.200.3908 | 858.552.1422 (fax) | www.cognella.com

Step-by-Step Business Math and Statistics

Jin W. Choi
DePaul University

cognella™
San Diego, CA

First published in the United States of America in 2011 by University Readers, Inc.

15 14 13 12 11 1 2 3 4 5

Printed in the United States of America

ISBN: 978-1-60927-872-4

cognella™

www.cognella.com 800.200.3908

Contents

Acknowledgments

I would like to thank many professors who had used this book in their classes. Especially, Professors Bala Batavia, Burhan Biner, Seth Epstein, Teresa Klier, Jin Man Lee, Norman Rosenstein, and Cemel Selcuk had used previous editions of this book in teaching GSB420 Applied Quantitative Analysis at DePaul University. Their comments and feedbacks were very useful in making this edition more user-friendly.

Also, I would like to thank many current and past DePaul University's Kellstadt Graduate School of Business MBA students who studied business mathematics and statistics using the framework laid out in this book. Their comments and feedbacks were equally important and useful in making this book an excellent guide into the often-challenging fields of mathematics and statistics. I hope and wish that the knowledge gained via this book would help them succeed in their business endeavors.

As is often the case with equations and numbers, I am sure this book still has some errors. If you find some, please let me know at jchoi@depaul.edu.

Best wishes to those who use this book.

Jin W. Choi, Ph.D.
Kellstadt Graduate School of Business
DePaul University
Chicago, IL 60604
jchoi@depaul.edu

Part 1. Business Mathematics

There are 4 chapters in this part of business mathematics: Algebra review, calculus review, optimization techniques, and economic applications of algebra and calculus.

Chapter 1. Algebra Review

A. The Number System

The number system is comprised of real numbers and imaginary numbers. Real numbers are, in turn, grouped into natural numbers, integers, rational numbers, and irrational numbers.

1. Real Numbers = numbers that we encounter everyday during a normal course of life → the numbers that are **real** to us.

   i. Natural numbers = the numbers that we often use to count items → counting trees, apples, bananas, etc.: 1, 2, 3, 4, …

      a. odd numbers: 1, 3, 5, …

      b. even numbers: 2, 4, 6, …

   ii. Integers = whole numbers without a decimal point: 0, ± 1, ± 2, ± 3, ± 4, ….

      a. positive integers: 1, 2, 3, 4, …

      b. negative integers: –1, –2, –3, –4, …

   iii. Rational numbers = numbers that can be expressed as a fraction of integers such as a/b (= a÷b) where both a and b are integers

      a. finite decimal fractions: 1/2, 2/5, etc.

      b. (recurring or periodic) infinite decimal fractions: 1/3, 2/9, etc.

   iv. Irrational Numbers = numbers that can NOT be expressed as a fraction of integers = nonrecurring infinite decimal fractions:

      a. n-th roots such as $\sqrt{2}, \sqrt[3]{5}, 7^{\sqrt{3}}$, etc.

      b. special values such as π (=pi), or e (=exponential), etc.

v. Undefined fractions:

a. any number that is divided by a zero such as k/0 where k is any number

b. a zero divided by a zero = 0/0

c. an infinity divided by an infinity = $\frac{\infty}{\infty}$

d. a zero divided by an infinity = $\frac{0}{\infty}$

vi. Defined fractions:

a. a one divided by a very small number → $\frac{1}{0.0000000001} = \frac{1}{10^{-10}} = 10^{10} = 10{,}000{,}000{,}000 \approx$ a very large number such as a number that can approach ∞

b. a one divided by a very large number → 1/(a large number) = a small number → $\frac{1}{\infty} \approx 0$

c. a scientific notion → the use of exponent

$2.345E+2 = 2.345 \times 10^2 = 234.5$

$2.345E+6 = 2.345 \times 10^6 = 2{,}345{,}000$

$2.345E–2 = 2.345 \times 10^{-2} = 2.345 \cdot \frac{1}{10^2} = 2.345 \cdot \frac{1}{100} = 0.02345$

$2.345E–6 = 2.345 \times 10^{-6} =$

$$2.345 \cdot \frac{1}{10^6} = 2.345 \cdot \frac{1}{1{,}000{,}000} = 0.000002345$$

Similarly, a caret (^) can be used as a sign for an exponent:

X^n = X^n → X^{10} = X^10

Note: **For example, E+6 means move the decimal point 6 digits to the right of the original decimal point whereas E-6 means move the decimal point 6 digits to the left of the original decimal point.**

2. Imaginary Numbers = numbers that are not easily encountered and recognized on a normal course of life and thus, not real enough (or imaginary) to an individual.
→ Often exists as a mathematical conception.

$i = \sqrt{-1}$ $\sqrt{-2} = \sqrt{2}i = i\sqrt{2}$

$\sqrt{-4} = 2i$ $(5i)^2 = -25$

B. Rules of Algebra

1.	$a + b = b + a$	→	$2 + 3 = 3 + 2$	→	5
2.	$ab = ba$	→	$2 \times 3 = 3 \times 2$	→	6
3.	$aa^{-1} = 1$ for $a \neq 0$	→	$2 \times 2^{-1} = 2^0 = 1$		
4.	$a(b + c) = ab + ac$	→	$2 \times (3 + 4) = 2 \times 3 + 2 \times 4$	→	14
5.	$a + (-a) = a - (+a) = 0$	→	$2 + (-2) = 2 - (+2) = 2 - 2 = 0$		
6.	$(-a)b = a(-b) = -ab$	→	$(-2) \times 3 = 2 \times (-3)$	→	–6
7.	$(-a)(-b) = ab$	→	$(-2) \times (-3) = 2 \times 3$	→	6
8.	$(a + b)^2 = a^2 + 2ab + b^2$	→	$(2 + 3)^2 = 2^2 + 2(2)(3) + 3^2$	→	25
9.	$(a - b)^2 = a^2 - 2ab + b^2$	→	$(2 - 3)^2 = 2^2 - 2(2)(3) + 3^2$	→	1
10.	$(a + b)(a - b) = a^2 - b^2$	→	$(2 + 3)(2 - 3) = 2^2 - 3^2$	→	–5
11.	$\frac{-a}{-b} = (-a)/(-b) = a/b$	→	$\frac{-2}{-3} = (-2)/(-3) = 2/3$	→	$\frac{2}{3}$
12.	$\frac{-a}{b} = \frac{a}{-b} = -1 \cdot \frac{a}{b} = -\frac{a}{b}$	→	$\frac{-2}{3} = (2)/(-3) = \frac{2}{-3}$	→	$-\frac{2}{3}$
13.	$a + \frac{b}{c} = \frac{ac + b}{c}$	→	$2 + \frac{3}{4} = \frac{2 \cdot 4 + 3}{4}$	→	$\frac{11}{4}$
14.	$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$	→	$\frac{2}{3} + \frac{4}{5} = \frac{2 \cdot 5 + 3 \cdot 4}{3 \cdot 5}$	→	$\frac{22}{15}$

15. $a \times \frac{b}{c} = a \cdot \frac{b}{c} = \frac{ab}{c}$ → $2 \times \frac{3}{4} = 2 \cdot \frac{3}{4} = \frac{2 \cdot 3}{4}$ → $\frac{6}{4}$

16. $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$ → $\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{2 \cdot 5}{3 \cdot 4}$ → $\frac{10}{12}$

17. $a^{1/2} = a^{0.5} = \sqrt{a}$ where $a \geq 0$ → $2^{1/2} = 2^{0.5} = \sqrt{2}$ → 1.4142

18. $a^{1/n} = \sqrt[n]{a}$ where $a \geq 0$ → $2^{1/3} = \sqrt[3]{2}$ → 1.2599

19. $\sqrt{ab} = \sqrt{a} * \sqrt{b}$ → $\sqrt{2 \cdot 3} = \sqrt{2} * \sqrt{3} = \sqrt{6}$ → 2.4495

20. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ → $\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$ → 0.8165

21. $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ → $\sqrt{2+3} \neq \sqrt{2} + \sqrt{3}$

22. $\frac{\sqrt{a}}{b} \neq \sqrt{\frac{a}{b}}$ → $\frac{\sqrt{2}}{3} \neq \sqrt{\frac{2}{3}}$

C. Properties of Exponents → Pay attention to equivalent notations

It is very important that we know the following properties of exponents:

1. $X^0 = 1$ → Note that 0^0 = undefined

2. $\frac{1}{X^b} = X^{-b} = X \wedge (-b)$ → $\frac{1}{X^{10}} = X^{-10} = X \wedge (-10)$

3. $X^a * X^b = X^a \cdot X^b = X^a X^b = X^{a+b}$ → $X \wedge (a+b)$

→ $X^2 * X^3 = X^2 \cdot X^3 = X^2 X^3 = X^{2+3} = X^5$

→ $2^3 \cdot 2^4 = 2^{3+4} = 2^7$ = 128

4. $(X^a)^b = X^{a*b} = X^{a \cdot b} = X^{ab}$ → $X \wedge ab$

→ $(X^2)^3 = X^{2*3} = X^{2 \cdot 3} = X^6$

→ $(2^3)^4 = 2^{3\cdot 4} = 2^{12} = 4096$

5. $\frac{X^a}{X^b} = X^a \cdot X^{-b} = X^{a-b}$

→ $\frac{X^2}{X^3} = X^2 \cdot X^{-3} = X^{2-3} = X^{-1} = \frac{1}{X}$

6. $(XY)^a = X^a * Y^a = X^a \cdot Y^a = X^a Y^a$

→ $(XY)^2 = X^2 * Y^2 = X^2 \cdot Y^2 = X^2 Y^2$

7. $\sqrt[n]{X} = X^{\frac{1}{n}} = X^{1/n}$

→ $\sqrt{4} = 4^{\frac{1}{2}} = 4^{1/2} = 4^{0.5} = (2^2)^{0.5} = 2^{2\cdot 0.5} = 2^1 = 2$

8. $X^{p/q} = (X^{1/q})^p = (X^p)^{1/q} = \sqrt[q]{X^p}$

→ $2^{10/5} = 2^2 = (2^{1/5})^{10} = (2^{10})^{1/5} = \sqrt[5]{2^{10}} = 2^2 = 4$

→ $8^{2/3} = (2^3)^{2/3} = 2^2 = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

D. Linear and Nonlinear Functions

1. Linear Functions

Linear Functions have the general form of:

Y = a + b X

where Y and X are variables and a and b are constants. More specifically, a is called an intercept and b, a slope coefficient. The most visually distinguishable character of a linear function is that it is a straight line. Note that +b means a positive slope and –b means a negative slope.

2. Nonlinear Functions

There are many different types of nonlinear functions such as polynomial, exponential, logarithmic, trigonometric functions, etc. Only polynomial, exponential and logarithmic functions will be briefly explained below.

i) The n-th degree polynomial functions have the following general form:

$$Y = a + bX + cX^2 + dX^3 + ... + pX^{n-1} + qX^n$$

Or alternatively expressed as:

$$Y = qX^n + pX^{n-1} + ... + dX^3 + cX^2 + bX + a$$

where a, b, c, d, …, p and q are all constant numbers called coefficients and n is the largest exponent value.

Note that the n-th degree polynomial function is named after the highest value of n. For example, when n = 2, it is most often called a quadratic function, instead of a second-degree polynomial function, and has the following form:

$$Y = a + bX + cX^2$$

When n = 3, it is called a third-degree polynomial function or a cubic function and has the following form:

$$Y = a + bX + cX^2 + dX^3$$

ii) Finding the Roots of a Polynomial Function

Often, it is important and necessary to find roots of a polynomial function, which can be a challenging task. An n-th degree polynomial function will have n roots. Thus, a third degree polynomial function will have 3 roots and a quadratic function, two roots. These roots need not be always different and in fact, can have the same value. Even though finding roots to higher-degree polynomial functions is difficult, the task of finding the roots of a quadratic equation is manageable if one relies on either the factoring method or the quadratic formula.

If we are to find the roots to a quadratic function of:

$$aX^2 + bX + c = 0$$

we can find their two roots by using the following quadratic formula:

$$X_1, X_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

iii) Examples:

Find the roots, X_1 and X_2, of the following quadratic equations:

(a) $X^2 - 3X + 2 = 0$

Factoring Method[1]:

$$X^2 - 3X + 2 = (X - 1)\cdot(X - 2) = 0$$

Therefore, we find two roots as: $X_1 = 1$ and $X_2 = 2$.

Quadratic Formula[2]:

Note: $a = 1$, $b = -3$, and $c = 2$

$$X_1, X_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(2)}}{2\cdot 1}$$

$$= \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = 1, 2$$

(b) $4X^2 + 24X + 36 = 0$

Factoring Method:

$$4X^2 + 24X + 36 = (2X + 6)\cdot(2X + 6) = (2X + 6)^2 = 4(X + 3)^2 = 0$$

Therefore, we find two identical roots (or double roots) as:

$$X_1 = X_2 = -3$$

Quadratic Formula:

Note: $a = 4$, $b = 24$, and $c = 36$

$$X_1, X_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(24) \pm \sqrt{(24)^2 - 4(4)(36)}}{2\cdot 4}$$

---

[1] The factoring method often seems more convenient for people with great experience with algebra. That is, the easiness comes with experience. Those who lack algebraic skill may be better off using the quadratic formula.

[2] In order to use the quadratic formula successfully, one must match up the values for a, b, and c correctly.

$$= \frac{-24 \pm \sqrt{576 - 576}}{8} = \frac{-24 \pm 0}{8} = -\frac{24}{8} = -3$$

(c) $4X^2 - 9Y^2 = 0$

Factoring Method:

$$4X^2 - 9Y^2 = (2X - 3Y) \cdot (2X + 3Y) = 0$$

Therefore, we find two roots as:

$$X_1 = \frac{3Y}{2} = 1.5Y \text{ and } X_2 = -\frac{3Y}{2} = -1.5Y$$

Quadratic Formula[3]:

Note: $a = 4$, $b = 0$, and $c = -9Y^2$

$$X_1, X_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(0) \pm \sqrt{(0)^2 - 4(4)(-9Y^2)}}{2 \cdot 4}$$

$$= \frac{0 \pm \sqrt{0 + 144Y^2}}{8} = \frac{\pm 12Y}{8} = \pm \frac{3}{2} Y = 1.5Y, -1.5Y$$

E. Exponential and Logarithmic Functions

1. Exponential Functions

An exponential function has the form of $Y = a \cdot b^X$ where a and b are constant numbers. The simplest form of an exponential function is $Y = b^X$ where b is called the base and X is called an exponent or a growth factor.

A unique case of an exponential function is observed when the base of e is used. That is, $Y = e^X$ where $e \approx 2.718281828$. Because this value of e is often identified with natural phenomena, it is called the “natural” base[4].

---

[3] One must be very cognizant of the construct of this quadratic equation. Because we are to find the roots associated with X, $-9Y^2$ should be considered as a constant term, like c in the quadratic equation.

[4] Technically, the expression $\left(1 + \frac{1}{n}\right)^n$ approaches e as n increases. That is, as n approaches $+\infty$, $e \approx 2.718281828$.

Examples>

In order to be familiar with how exponential functions work, please verify the following equalities by using a calculator.

a. $5e^2 \cdot e^4 = 5e^{2+4} = 5e^6 = 5 \cdot 403.4287935 = 2017.143967$

b. $(5e^3) \cdot (3e^4) = 15e^7 = 15 \cdot 1096.633158 = 16449.49738$

c. $10e^3 \div 2e^4 = \frac{10}{2} \cdot e^{3-4} = 5e^{-1} = \frac{5}{e} = \frac{5}{2.718281828} = 1.839397206$

2. Logarithmic Functions

The logarithm of Y with base b is denoted as "$\log_b Y$" and is defined as:

$$\log_b Y = X \text{ if and only if } b^X = Y$$

provided that b and Y are positive numbers with $b \neq 1$. The logarithm enables one to find the value of X given $2^X = 4$ or $5^X = 25$. In both of these cases, we can easily find X=2 due to the simple squaring process involved. However, finding X in $2^X = 5$ is not easy. This is when knowing a logarithm comes in handy.

Examples>

Convert the following logarithmic functions into exponential functions:

$\log_2 8 = X$ → $2^X = 8$ → X = 3

$\log_5 1 = 0$ → $5^0 = 1$

$\log_4 4 = 1$ → $4^1 = 4$

$\log_{1/2} 4 = -2$ → $\left(\frac{1}{2}\right)^{-2} = \left(2^{-1}\right)^{-2} = 2^{+2} = 2^2 = 4$

a. Special Logarithms: A common logarithm and a natural logarithm.

i) A Common Logarithm = a logarithm with base 10 and often denoted without the base value.

That is, $\log_{10} X = \log X$ → read as "a (common) logarithm of X."

ii) A Natural Logarithm = a logarithm with base e and often denoted as 'ln".

That is, $\log_e X = \ln X$ → read as "a natural logarithm of X."

b. Properties of Logarithms

i) Product Property: $\log_b mn = \log_b m + \log_b n$

ii) Quotient Property: $\log_b \frac{m}{n} = \log_b m - \log_b n$

iii) Power Property: $\log_b m^n = n \cdot \log_b m$

Example 1> Using the above 3 properties of logarithm, verify the following equality or inequality by using a calculator.

i) $\ln 30 = \ln(5 \cdot 6) = \ln 5 + \ln 6$ → 3.401197

ii) $\ln \frac{20}{40} = \ln 20 - \ln 40 = \ln 0.5$ → –0.693147

iii) $\frac{\ln 20}{\ln 40} \neq \ln \frac{20}{40}$

iv) $\ln 10^3 = 3 \cdot \ln 10 = \ln 1000$ → 6.907755

Example 2> Find X in $2^X = 5$. (This solution method is a bit advanced.)

In order to find X,

(1) we can take a natural (or common) logarithm of both sides as:
$\ln 2^X = \ln 5$
(2) rewrite the above as: $X \cdot \ln 2 = \ln 5$ by using the Power Property
(3) solve for X as: $X = \frac{\ln 5}{\ln 2}$
(4) use the calculator to find the value of X as:
$$X = \frac{\ln 5}{\ln 2} = \frac{1.6094379}{0.6931471} = 2.321928095$$

Additional topics of exponential and logarithmic functions are complicated and require many additional hours of study. Because it is beyond our realm, no additional attempt to explore this topic is made herein[5].

F. Useful Mathematical Operators

1. Summation Operator = Sigma = Σ → $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i = \sum X_i$

$\sum_{i=1}^{n} X_i = X_1 + X_2 + ... + X_{n-1} + X_n$ = Sum X_i's where i goes from 1 to n.

Examples: Given the following X data, verify the summation operation.

i =	1	2	3	4	5
X_i =	25	19	6	27	23

a. $\sum_{i=1}^{3} X_i = X_1 + X_2 + X_3 = 25 + 19 + 6 = 50$

b. $\sum_{i=1}^{5} X_i = X_1 + X_2 + X_3 + X_4 + X_5 = 25 + 19 + 6 + 27 + 23 = 100$

c. $\sum_{i=3}^{5} X_i = X_3 + X_4 + X_5 = 6 + 27 + 23 = 56$

d. $\sum_{i=1}^{3} X_i + \sum_{i=3}^{5} X_i = (X_1 + X_2 + X_3) + (X_3 + X_4 + X_5)$
$= (25 + 19 + 6) + (6 + 27 + 23) = 50 + 56 = 106$

e. $\sum_{i=1}^{3} X_i - \sum_{i=3}^{5} X_i = (X_1 + X_2 + X_3) - (X_3 + X_4 + X_5)$
$= (25 + 19 + 6) - (6 + 27 + 23) = 50 - 56 = -6$

2. Multiplication Operator = pi = Π → $\prod_{i=1}^{n} X_i = \prod X_i$

$\prod_{i=1}^{n} X_i = X_1 \cdot X_2 \cdot .... \cdot X_{n-1} \cdot X_n$ = Multiply X_i's where i goes from 1 to n.

[5] For detailed discussions and examples on this topic, please consult high school algebra books such as Algebra 2, by Larson, Boswell, Kanold, and Stiff. ISBN=13:978-0-618-59541-9.

Examples: Given the following X data, verify the multiplication operation.

i =	1	2	3	4	5
X_i =	3	5	6	2	4

a.

$$\prod_{i=1}^{3} X_i = X_1 \cdot X_2 \cdot X_3 = 3 \cdot 5 \cdot 6 = 90$$

b. $$\prod_{i=1}^{5} X_i = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5 = 3 \cdot 5 \cdot 6 \cdot 2 \cdot 4 = 720$$

c. $$\prod_{i=3}^{5} X_i = X_3 \cdot X_4 \cdot X_5 = 6 \cdot 2 \cdot 4 = 48$$

d. $$\prod_{i=1}^{3} X_i + \prod_{i=3}^{5} X_i = (X_1 \cdot X_2 \cdot X_3) + (X_3 \cdot X_4 \cdot X_5)$$
$$= (3 \cdot 5 \cdot 6) + (6 \cdot 2 \cdot 4) = 90 + 48 = 138$$

e. $$\prod_{i=1}^{3} X_i - \prod_{i=4}^{5} X_i = (X_1 \cdot X_2 \cdot X_3) - (X_4 \cdot X_5)$$
$$= (3 \cdot 5 \cdot 6) - (2 \cdot 4) = 90 - 8 = 72$$

f. $$\sum_{i=1}^{2} X_i - \prod_{3}^{5} X_i = (X_1 + X_2) - (X_3 \cdot X_4 \cdot X_5)$$
$$= (3 + 5) - (6 \cdot 2 \cdot 4) = 8 - 48 = 40$$

G. Multiple-Choice Problems for Exponents, Logarithms, and Mathematical Operators:

Identify all equivalent mathematical expressions as correct answers.

1. $(X + Y)^2 =$

a. $X^2 + 2XY + Y^2$

b. $X^2 – 2XY + Y^2$

c. $X^2 + XY + Y^2$

d. $X^2 + 2XY + 2Y^2$

e. none of the above

2. $(X - Y)^2 =$

a. $X^2 + 2XY + Y^2$

b. $(X - Y)(X - Y)$

c. $X^2 - 2XY + Y^2$

d. $X^2 - XY + Y^2$

e. only (b) and (c) of the above

3. $(2X + 3Y)^2 =$

a. $4X^2 + 6YX + 9Y^2$

b. $4X^2 + 12XY + 9Y^2$

c. $2X^2 + 6XY + 3Y^2$

d. $4X^2 + 9Y^2$

e. none of the above

4. $(2X - 3Y)^2 =$

a. $4X^2 - 9Y^2$

b. $2X^2 + 6XY + 3Y^2$

c. $4X^2 - 12XY + 9Y^2$

d. $4X^2 + 9Y^2$

e. none of the above

5. $(2X^3)(6X^{10}) =$

a. $2X^{3+10}$

b. $12X^{30}$

c. $48X^{3/10}$

d. $12X^{13}$

e. none of the above

6. $(12X^6Y^2)(2Y^3X^2)(3X^3Y^4) =$

a. $72X^{11}Y^9$

b. $72X^{12}Y^8$

c. $17X^{10}Y^{10}$

d. $72Y^8 X^{12}$

e. only (b) and (d) of the above

7. $X^2(X + Y)^2 =$

a. $X^2(X^2 + 2XY + Y^2)$

b. $X^{2+2} + 2X^{1+2}Y + X^2Y^2$

c. $X^4 + 2X^3Y + X^2Y^2$

d. all of the above

e. none of the above

8. $\frac{X^3}{2} \cdot \frac{6}{X^2} =$

a. $3X^5$

b. $3X$

c. $3X^{-1}$

d. $12X$

e. none of the above

9. $(2X^3)/(6X^{10}) =$

a. $0.33333333X^{-7}$

b. $\frac{1}{3X^7}$

c. $\frac{1}{3}X^{-7}$

d. only (a) and (c) of the above

e. all of the above

10. $\frac{10}{X^9Y^5} \cdot X^5Y^3 =$

a. $10X^{-4}Y^{-2}$

b. $\frac{10}{X^4Y^2}$

c. $10X^9Y^5X^{-5}Y^{-3}$

d. only (a) and (b) of the above

e. all of the above

11. $24X^{0.5}Y^{1.5} \div 12X^{1.5}Y^{0.5} =$

a. $\frac{2Y}{X}$

b. $\frac{2X}{Y}$

c. $\frac{Y}{2X}$

d. $\frac{Y}{X}$

e. $\frac{X}{Y}$

12. $(64X)^{\frac{1}{2}}(8Y)^{\frac{1}{3}} \div 8X^{2.5}Y^{\frac{4}{3}} =$

a. $\frac{2}{X^2Y}$ b. $\frac{2}{XY}$ c. $\frac{2}{Y^2X}$

d. $\frac{2Y}{X^2}$ e. none of the above

13. $(2X^2)^3 =$

a. $2X^6$ b. $8X^6$ c. $8X^5$

d. $16X^6$ e. $(2X^3)^2$

14. $[(3X^4Y^3)^2]^2 =$

a. $9X^8Y^7$ b. $9X^{16}Y^{12}$ c. $81X^{16}Y^{10}$

d. $81X^{16}Y^{12}$ e. $81X^8Y^7$

15. $(4X^4Y^3)^2/(2X^2Y^2)^4 =$

a. Y^{-3} b. Y^2 c. $1/Y^2$

d. X^2 e. $1/X^2$

Using the following data, answer Problems 16 – 20.

i =	1	2	3	4	5	6	7
X_i =	30	52	67	22	16	42	34

16. $\sum_{i=1}^{3} X_i =$

a. 6 b. 140 c. 149
d. 104520 e. none of the above

17. $\prod_{i=4}^{6} X_i =$

a.	6	b.	80	c.	120
d.	14784	e.	none of the above		

18. $\sum_{i=1}^{2} X_i - \prod_{3}^{5} X_i =$

a.	– 23	b.	0	c.	2352
d.	2350	e.	none of the above		

19. $\sum_{i=5}^{7} X_i + \prod_{3}^{5} X_i =$

a.	92	b.	23584	c.	23676
d.	46432	e.	none of the above		

20. $\prod_{i=6}^{7} X_i + \sum_{i=3}^{4} X_i - \prod_{5}^{6} X_i =$

a.	0	b.	1428	c.	89
d.	672	e.	84		

21. Find the value of X in $3^X = 59049$.

a.	20	b.	15	c.	10
d.	5	e.	none of the above		

22. Identify the correct relationship(s) shown below:

a.	$X \log 20 = \log_{20} X$	b.	$15^X \neq 3^X \cdot 5^X$
c.	$5 \ln \frac{2}{5} = \ln 2$	d.	$\frac{\ln X}{\ln Y} = \ln X - \ln Y$
e.	none of the above is correct.		

Answers to Exercise Problems for Exponents and Mathematical Operators

1. $(X + Y)^2 =$

a.* $X^2 + 2XY + Y^2$ because
$(X + Y)(X + Y) = X^2 + XY + YX + Y^2 = X^2 + 2XY + Y^2$

2. $(X - Y)^2 =$

e.* only (b) and (c) of the above because
$(X - Y)(X - Y) = X^2 - XY - YX + Y^2 = X^2 - 2XY + Y^2$

3. $(2X + 3Y)^2 =$

b.* $4X^2 + 12XY + 9Y^2$ because
$(2X + 3Y)(2X + 3Y) = 4X^2 + 6XY + 6YX + 9Y^2$
$= 4X^2 + 12XY + 9Y^2$

4. $(2X - 3Y)^2 =$

c.* $4X^2 - 12XY + 9Y^2$ because
$(2X - 3Y)^2 = (2X - 3Y)(2X - 3Y) = 4X^2 - 6XY - 6YX + 9Y^2$
$= 4X^2 - 12XY + 9Y^2$

5. $(2X^3)(6X^{10}) =$

d.* $12X^{13}$ because $(2)(6)X^{3+10} = 12X^{3+10} = 12X^{13}$

6. $(12X^6Y^2)(2Y^3X^2)(3X^3Y^4) =$

a.* $72X^{11}Y^9$ because $(12)(2)(3)X^{6+2+3}Y^{2+3+4} = 72X^{11}Y^9$

7. $X^2(X + Y)^2 =$

d.* all of the above because
$X^2(X^2 + 2XY + Y^2) = X^{2+2} + 2X^{1+2}Y + X^2Y^2 = X^4 + 2X^3Y + X^2Y^2$

8. $\frac{X^3}{2} \cdot \frac{6}{X^2} =$

b.* $3X$ because $\frac{6X^3}{2X^2} = 3X^{3-2} = 3X$

9. $(2X^3)/(6X^{10}) =$

e.* all of the above because $(2/6)X^{3-10} = (1/3)X^{-7} = 0.3333X^{-7} = \frac{1}{3X^7}$

10. $\frac{10}{X^9Y^5} \cdot X^5Y^3 =$

d.* only (a) and (b) of the above because

$$10X^{-9}Y^{-5}X^{5}Y^{3}=10X^{-9+5}Y^{-5+3}=10X^{-4}Y^{-2}=\frac{10}{X^{4}Y^{2}}$$

11. $24X^{0.5}Y^{1.5}\div 12X^{1.5}Y^{0.5}=$

a.* $\frac{2Y}{X}$ because $\frac{24X^{0.5}Y^{1.5}}{12X^{1.5}Y^{0.5}}=2X^{0.5-1.5}Y^{1.5-0.5}=2X^{-1}Y^{1}=\frac{2Y}{X}$

12. $(64X)^{\frac{1}{2}}(8Y)^{\frac{1}{3}}\div 8X^{2.5}Y^{\frac{4}{3}}=$

a.* $\frac{2}{X^{2}Y}$ because $\frac{(64X)^{\frac{1}{2}}(8Y)^{\frac{1}{3}}}{8X^{2.5}Y^{\frac{4}{3}}}=\frac{(64)^{\frac{1}{2}}X^{\frac{1}{2}}(8)^{\frac{1}{3}}Y^{\frac{1}{3}}}{8X^{2.5}Y^{\frac{4}{3}}}$

$$=\frac{(8)X^{\frac{1}{2}}(2)Y^{\frac{1}{3}}}{8X^{2.5}Y^{\frac{4}{3}}}=X^{\frac{1}{2}-2.5}(2)Y^{\frac{1}{3}-\frac{4}{3}}=2X^{-2}Y^{-1}=\frac{2}{X^{2}Y}$$

13. $(2X^2)^3 =$

b.* $8X^6$ because $(2X^2)(2X^2)(2X^2) = (2)^3X^{2+2+2} = 2^3X^{2x3} = 8X^6$

14. $[(3X^4Y^3)^2]^2 =$

d.* $81X^{16}Y^{12}$ because $[3^2X^{4x2}Y^{3x2}]^2 = 3^{2x2}X^{8x2}Y^{6x2} = 81X^{16}Y^{12}$

15. $(4X^4Y^3)^2/(2X^2Y^2)^4 =$

c.* $1/Y^2$ because

$(4X^4Y^3)^2(2X^2Y^2)^{-4} = [(2^2)^2X^8Y^6][(2)^{-4}X^{-8}Y^{-8}] = Y^{-2} = 1/Y^2$

16. $\sum_{i=1}^{3}X_i =$

c.* 149 because $\sum_{i=1}^{3}X_i = X_1 + X_2 + X_3 = 30 + 52 + 67 = 149$

17. $\prod_{i=4}^{6}X_i =$

d.* 14784 because $\prod_{i=4}^{6}X_i = X_4 \cdot X_5 \cdot X_6 = 22 \cdot 16 \cdot 42 = 14784$

18. $\sum_{i=1}^{2} X_i - \prod_{i=3}^{5} X_i =$

e.* none of the above

because $\sum_{i=1}^{2} X_i - \prod_{i=3}^{5} X_i = (X_1 + X_2) - (X_3 \cdot X_4 \cdot X_5)$

$= (30 + 52) - (67 \cdot 22 \cdot 16) = 82 - 23584 = 23502$

19. $\sum_{i=5}^{7} X_i + \prod_{i=3}^{5} X_i =$

c.* 23676

because $\sum_{i=5}^{7} X_i + \prod_{i=3}^{5} X_i = (X_5 + X_6 + X_7) + (X_3 \cdot X_4 \cdot X_5)$

$= (16 + 42 + 34) + (67 \cdot 22 \cdot 16) = 92 + 23584 = 23676$

20 $\prod_{i=6}^{7} X_i + \sum_{i=3}^{4} X_i - \prod_{i=5}^{6} X_i =$ 20.

e.* 845

because $\prod_{i=6}^{7} X_i + \sum_{i=3}^{4} X_i - \prod_{i=5}^{6} X_i = (X_6 + X_7) + (X_3 + X_4) - (X_5 \cdot X_6)$

$= (42 \cdot 34) + (67 + 22) - (16 \cdot 42) = 1428 + 89 - 672 = 845$

21. Find the value of X in $3^X = 59049$.

c.* 10

In order to find X,

(1) we can take a natural (or common) logarithm of both sides as:

$$\ln 3^X = \ln 59049$$

(2) rewrite the above as: $X \cdot \ln 3 = \ln 59049$ by using the Power Property

(3) solve for X as: $X = \dfrac{\ln 59049}{\ln 3}$

(4) use the calculator to find the value of X as:

$$X = \frac{\ln 59049}{\ln 3} = \frac{10.9861}{1.09861} = 10$$

22. Identify the correct relationship(s) shown below:

e.* none of the above is correct.

Note that

a. $X \log 20 = \log 20^X \neq \log_{20} X$ b. $15^X = (3 \cdot 5)^X = 3^X \cdot 5^X$

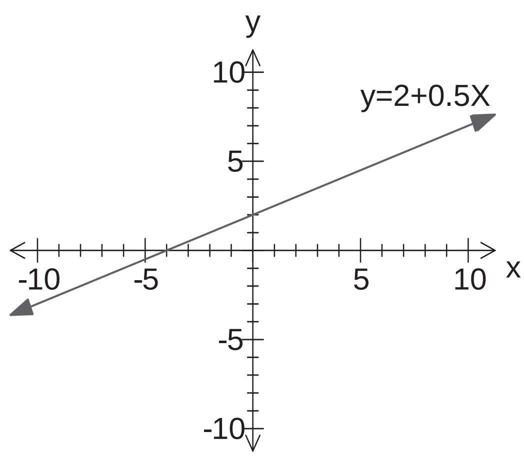
c. $5 \ln \frac{2}{5} = \ln(\frac{2}{5})^5 \neq \ln 2$ d. $\frac{\ln X}{\ln Y} \neq \ln \frac{X}{Y} = \ln X - \ln Y$

H. Graphs

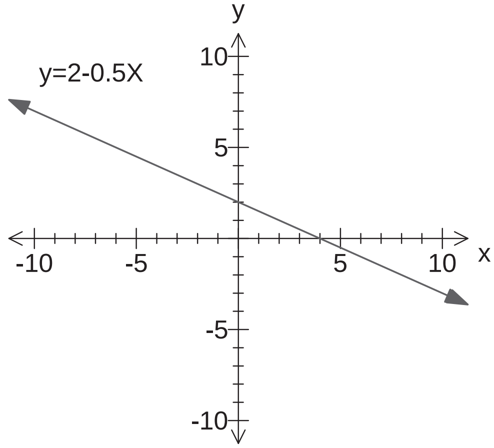
In economics and other business disciplines, graphs and tables are often used to describe a relationship between two variables – X and Y. X is often represented on a horizontal axis and Y, a vertical axis.

1. A Positive-Sloping Line and a Negative-Sloping Line

For example, a function of Y = 2 + 0.5X, as plotted below, has an intercept of 2 and a positive slope of +0.5. Therefore, it rises to the right (and declines to the left) and thus, is characterized as a positive sloping or upward sloping line. It shows a pattern where as X increases (decreases), Y increases (decreases). This relationship is also known as a direct relationship.



On the other hand, a function of Y = 2 – 0.5X as plotted below, has an intercept of 2 and a negative slope of –0.5. Therefore, it declines to the right (and rises to the left) and thus, is characterized as a negative sloping or downward sloping line. It shows a pattern where as X increases (decreases), Y decreases (increases). That is, because X and Y move in an opposite direction, it is also known as an indirect or inverse relationship.

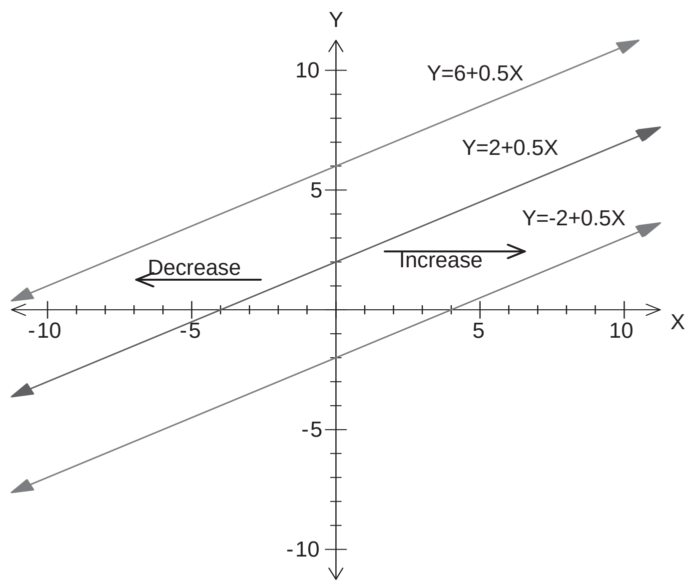


2. Shifts in the Lines

Often, the line can move up or down as the value of the intercept changes, while maintaining the same slope value. When the following two equations are plotted in addition to the original one we plotted above, we can see how the two lines differ from the original one by their respective intercept values:

Original Line: Y = 2 + 0.5X ← The middle line

New Line #1: Y = 6 + 0.5X ← The top line
New Line #2: Y = –2 + 0.5X ← The bottom line



Note 1: As the intercept term increases from 2 to 6, the middle line moves up to become the top line. This upward shift in the line indicates that the value of X has decreased while the Y value was held constant (or unchanged). Thus, the upward shift is the same as a shift to the left and indicates a decrease in X given the unchanged (or same) value of Y.

Note 2: As the intercept term decreases from 2 to –2, the middle line moves down to become the bottom line. This downward shift in the line indicates that the value of X has increased while the Y value was held constant (or unchanged). Thus, the downward shift is the same as a shift to the right and indicates an increase in X given the unchanged (or same) value of Y.

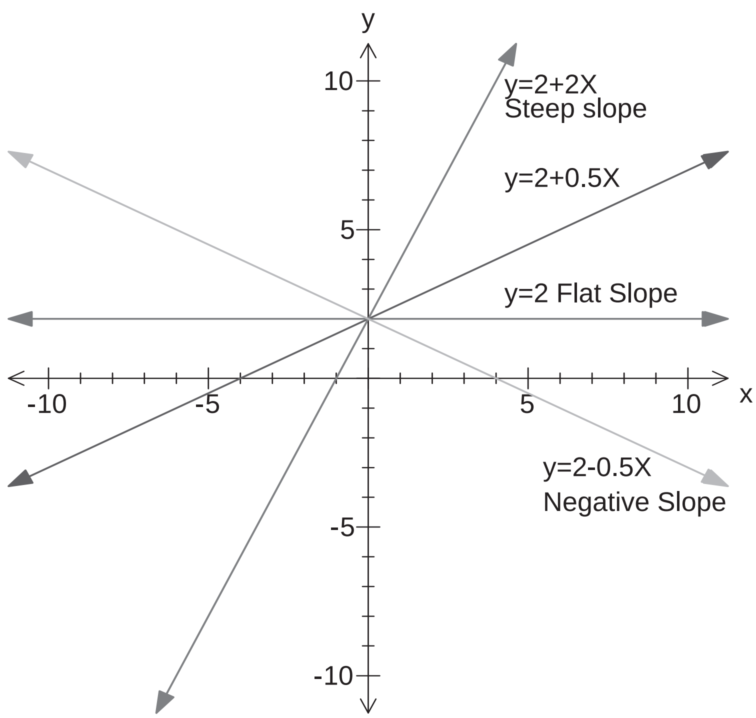
Note 3: This observation is often utilized in the demand and supply analysis of economics as a shift in the curve. A leftward shift is a "decrease" and a rightward shift is an "increase."

3. Changes in the Slope

When the value of a slope changes, holding the intercept unchanged, we will note that the line will rotate around the intercept as the center. Let's plot two new lines in addition to the original line as follows:

Original Line: $Y = 2 + 0.5X$ ← The original (=middle) line

New Line #1: $Y = 2 + 2X$ ← The top line
New Line #2: $Y = 2 + 0X = 2$ ← The flat line
New Line #3: $Y = 2 – 0.5X$ ← The bottom line



Note that the steepness (or flatness) of the slope as the value of the slope changes. Likewise, note the relationship among a flat, a positive, and a negative slope.

I. Applications: Compound Interest

1. The Concept of Periodic Interest Rates

Assume that the annual percentage rate (APR) is $(r \cdot 100)$%. That is, if an APR is 10%, then r = 0.1. Also, define FV = future value, PV = present value, and t = number of years to a maturity.

i) Annual compounding for t years → $FV = PV \cdot (1+r)^t$

ii) Semiannual compounding for t years → $FV = PV \cdot (1+\frac{r}{2})^{2t}$

iii) Quarterly compounding for t years → $FV = PV \cdot (1+\frac{r}{4})^{4t}$

iv) Monthly compounding for t years → $FV = PV \cdot (1+\frac{r}{12})^{12t}$

v) Weekly compounding for t years → $FV = PV \cdot (1+\frac{r}{52})^{52t}$

vi) Daily compounding for t years → $FV = PV \cdot (1+\frac{r}{365})^{365t}$

vii) Continuous compounding for t years[6] → $FV = PV \cdot e^{rt}$

Examples>

Assume that $100 is deposited at an annual percentage rate (APR) of 12% for 1 year.

i) Annual compounding → one 1-year deposit → 1 interest calculation

$$FV = PV \cdot (1+r)^t = \$100 \cdot (1+0.12)^1 = \$112.00$$

ii) Semiannual compounding →two ½-year deposits → 2 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{2})^{2t} = \$100 \cdot (1+\frac{0.12}{2})^{2\cdot 1} = \$100 \cdot (1+0.06)^2 = \$112.36$$

iii) Quarterly compounding →four ¼-year deposits → 4 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{4})^{4t} = \$100 \cdot (1+\frac{0.12}{4})^{4\cdot 1} = \$100 \cdot (1+0.03)^4 = \$112.55$$

[6] Do you remember that this is an exponential function with a natural base of e?

iv) Monthly compounding → twelve 1/12-year deposits → 12 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{12})^{12t} = \$100 \cdot (1+\frac{0.12}{12})^{12 \cdot 1} = \$100 \cdot (1+0.01)^{12} = \$112.68$$

v) Weekly compounding →fifty-two 1/52-year deposits → 52 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{52})^{52t} = \$100 \cdot (1+\frac{0.12}{52})^{52 \cdot 1} = \$100 \cdot (1+0.0023077)^{52} = \$112.73$$

vi) Daily compounding →365 1/365-year deposits → 365 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{365})^{365t} = \$100 \cdot (1+\frac{0.12}{365})^{365 \cdot 1} = \$100 \cdot (1+0.000328767)^{365} = \$112.74$$

vii) Continuous compounding for 1 year → continuous interest calculations

$$FV = PV \cdot e^{rt} = \$100 \cdot e^{0.12 \cdot 1} = \$100 \cdot e^{0.12} = \$112.75$$

Examples>

Calculate the annual rate of return (ROR) based on the various compounding schemes shown above.

i) For annual compounding,

$$\text{ROR} = \frac{P_1 - P_0}{P_0} = \frac{112-100}{100} = 0.12 \rightarrow 12\%$$

ii) For semi-annual compounding,

$$\text{ROR} = \frac{P_1 - P_0}{P_0} = \frac{112.36-100}{100} = 0.1236 \rightarrow 12.36\%$$

iii) For monthly compounding,

$$\text{ROR} = \frac{P_1 - P_0}{P_0} = \frac{112.68 - 100}{100} = 0.1268 \rightarrow 12.68\%$$

Note: The rate of return on an annual basis is known as the Annual Percentage Yield (APY). Even though APR may be the same, APY will increase as the frequency of compounding increases → because an interest is earned on an interest more frequently.

2. Annuity Calculation

Annuity Formulas:

$$FV = \frac{A \cdot [(1+i)^n - 1]}{i}$$

$$PV = \frac{A \cdot [(1+i)^n - 1]}{i \cdot (1+i)^n}$$

where A= the fixed annuity amount; n = the number of periods; and i = a periodic interest rate. Of course, FV = the future (or final or terminal) value and PV = the present (or current) value.

Examples

i) If you obtain a 30 year mortgage loan of $100,000 at an annual percentage rate (APR) of 6%, what would be your monthly payment?

Answer:

$$100{,}000 = \frac{A \cdot [(1+\frac{0.06}{12})^{12x30} - 1]}{\frac{0.06}{12} \cdot (1+\frac{0.06}{12})^{12x30}}$$

Therefore, A=$599.55

ii) If you invest $1,000 a month in an account that is guaranteed to yield a 10% rate of return per year for 30 years (with a monthly compounding), what will be the balance at the end of the 30-year period?

Answer:

$$FV = \frac{\$1{,}000 \cdot [(1+\frac{0.1}{12})^{360} - 1]}{\frac{0.1}{12}} = \$2{,}260{,}487.92$$

iii) If you are guaranteed of a 10% rate of return for 30 years, how much should you save and invest each month to accumulate $1 million at the end of the 30-year period?

Answer:

$$\$1{,}000{,}000 = \frac{A \cdot [(1+\frac{0.1}{12})^{360} - 1]}{\frac{0.1}{12}}$$

Therefore, A=$442.38

iv) Suppose that you have saved up $100,000 for your retirement. You expect that you can continuously earn 10% each year for your $100,000. If you know that you are going to live for 15 additional years from the date of your retirement and that the balance of your retirement fund will be zero at the end of the 15-year period, how much can you withdraw to spend each month?

Answer:

$$\$100{,}000 = \frac{A \cdot [(1+\frac{0.1}{12})^{180} - 1]}{\frac{0.1}{12} \cdot (1+\frac{0.1}{12})^{180}}$$

Therefore, A=$1,074.61

v) Assume the same situation as Problem 4 above, except that now you have to incorporate an annual inflation rate of 3%. What will be the possible monthly withdrawal, net of inflation?

Answer:

$$\$100{,}000 = \frac{A \cdot [(1+\frac{0.1-0.03}{12})^{180} - 1]}{\frac{0.1-0.03}{12} \cdot (1+\frac{0.1-0.03}{12})^{180}}$$

Therefore, A=$898.83

Note: Combining Answers to Problems (iv) and (v) above, it means that you will be actually withdrawing $1,074.61 per month but its purchasing power will be equivalent to $898.83. This is because inflation only erodes the purchasing power; it does not reduce the actual amount received. If one goes through a professional financial planning, the financial planner will expand on this simple assumption to a more complex and realistic scenario.

vi) Assuming only annual compounding, how long will it take to double your investment if you earn 10% per year?

Answer[7]:

$$A \cdot (1+0.1)^{x} = 2A$$

$$\therefore 1.1^{x} = 2$$

Now, take the natural logarithm of both sides as follows:

$$\ln 1.1^{x} = \ln 2$$

$$X \cdot \ln 1.1 = \ln 2$$

$$\therefore X = \frac{\ln 2}{\ln 1.1} = 7.2725\,years$$

vii) Assuming monthly compounding, how long will it take to double your investment if you earn 10% per year?

Answer:

$$A \cdot (1+\frac{0.1}{12})^{x} = 2A$$

$$1.0083333^{x} = 2$$

$$\ln 1.0083333^{x} = \ln 2$$

$$X \cdot \ln 1.0083333 = \ln 2$$

$$\therefore X = \frac{\ln 2}{\ln 1.0083333} = 83.5\,months = 6.96\,years$$

[7] When either the natural logarithm or the common logarithm is taken, the exponent, X, as in this case, will become a coefficient as shown herein. Then, use the calculator with a "ln" function to complete the calculation.

viii) Assume that you have a 30-year, $100,000 mortgage loan at an annual percentage rate (APR) of 6%. How long will it take you to pay off this loan if you pay off $1,000 a month?

Answer: Use the information on Answers to Problem 1 as follows:

$$100{,}000 = \frac{1{,}000 \cdot [(1+\frac{0.06}{12})^X - 1]}{\frac{0.06}{12} \cdot (1+\frac{0.06}{12})^X}$$

Therefore,

$$100{,}000 \cdot \frac{0.06}{12} \cdot (1+\frac{0.06}{12})^X = 1{,}000 \cdot [(1+\frac{0.06}{12})^X - 1]$$

$$500 \cdot (1+\frac{0.06}{12})^X = 1000 \cdot (1+\frac{0.06}{12})^X - 1000$$

$$500 \cdot (1.005)^X = 1000$$

$$X \cdot \ln 1.005 = \ln 2$$

$$X = \frac{\ln 2}{\ln 1.005} = 138.975 \textit{months} = 11.58 \textit{years}$$

J. Inequalities

1. If a > 0 and b > 0, then (a+b) > 0 and ab > 0

   If a=7 and b=5, then (7+5) > 0 and (7)(5) > 0

2. If a > b, then (a–b) > 0

   If a=7 and b=5, then (7–5) > 0

3. If a > b, then (a+c) > (b+c) for all c

   If a=7 and b=5, then (7+c) > (5+c) → 7 > 5

4. If a > b and c > 0, then ac > bc

   If a=7 and b=5 and c=3, then (7)(3) > (5)(3) → 21 > 15

5. If $a > b$ and $c < 0$, then $ac < bc$

If $a=7$ and $b=5$ and $c= -3$, then $(7)(-3) < (5)(-3) \rightarrow -21 < -15$

K. Absolute Values and Intervals

1. $|X| = X$ if $X \geq 0$ and $|X| = -X$ if $X \leq 0$

Examples>

a. $|+5| = +5 = 5$ and $|-5| = -(-5) = +5 = 5$

b. $|+10| = 10$ and $|-10| = -(-10) = 10$

2. If $|X| \leq n$, then $-n \leq X \leq n$

Examples>

a. If $|X| \leq 5$, then $-5 \leq X \leq 5$

b. If $|X-2| \leq 5$, then $-5 \leq X-2 \leq 5 \rightarrow -5+2 \leq X \leq 5+2$

$\rightarrow -3 \leq X \leq 7$

c. If $|2X+4| < 10$, then $-10 < 2X+4 < 10 \rightarrow -14 < 2X < 6$

$\rightarrow -7 < X < 3$

3. If $|X| > n$, then $X > n$ if $X > 0$ or $-X > n$ if $X < 0$

Note that when a negative number is multiplied to both sides of the inequality sign, the direction of the inequality sign reverses.

Examples>

a. If $|X| > 5$,
then, $X > 5$ or $-X > 5 \rightarrow X < -5$

b. If $|X-3| > 5$,
then, $(X-3) > 5 \rightarrow X > 8$

or $-(X-3) > 5 \rightarrow (X-3) < -5 \rightarrow X < -2$

c. If $|6 - 3X| > 12$,

then, $(6 - 3X) > 12 \rightarrow -3X > 6 \rightarrow X < -2$

or

$-(6 - 3X) > 12 \rightarrow (6 - 3X) < -12 \rightarrow -3X < -18 \rightarrow X > 6$

L. A System of Linear Equations in Two Unknowns

Given the following system of linear equations, solve for X and Y.

$$3X + 2Y = 13$$
$$4Y - 2X = 2$$

1. Solution Method 1: The Substitution Method

(1) Rearrange the bottom equation for X as follows:

$$2X = 4Y - 2 \quad \rightarrow \quad X = 2Y - 1$$

(2) Substitute this X into the top equation as follows:

$$3(2Y - 1) + 2Y = 13 \quad \rightarrow \quad 8Y = 16 \quad \rightarrow \quad Y = 2$$

(3) Substitute this Y into any of the above equation for X value:

$$X = 2Y - 1 = 2 \times 2 - 1 = 3$$

(4) Verify if the values of X and Y satisfy the system of equations:

$$3X + 2Y = 3 \times 3 + 2 \times 2 = 13$$
$$4Y - 2X = 4 \times 2 - 2 \times 3 = 2$$

(5) Verification completed and solutions found.

2. Solution Method 2: The Elimination Method

(1) Match up the variables as follows:

$$3X + 2Y = 13$$
$$-2X + 4Y = 2$$

(2) Multiply either of the two equations to find a common coefficient. (Y is chosen to be eliminated and thus, the top equation is multiplied by 2 as follows:)

$$2 \times 3X + 2 \times 2Y = 2 \times 13 \quad \rightarrow \quad 6X + 4Y = 26$$

(3) Subtract the bottom equation from the adjusted top equation in (2) above and obtain:

$$\begin{array}{c} 6X + 4Y = 26 \\ -(-2X + 4Y = 2) \\ \hline \end{array} \quad \text{or} \quad \begin{array}{c} 6X + 4Y = 26 \\ 2X - 4Y = -2 \\ \hline \end{array}$$

$$6X - (-2X) + 4Y - 4Y = 26 - 2 \quad \text{or} \quad 6X + 2X + 4Y - 4Y = 26 - 2$$
$$8X = 24 \qquad\qquad 8X = 24$$
$$X = 3 \qquad\qquad X = 3$$

(4) Substitute this X into any of the above equation for Y value:

$$6X + 4Y = 26$$
$$6 \times 3 + 4Y = 26$$
$$4Y = 26 - 18$$
$$Y = 2$$

(5) Verify if the values of X and Y satisfy the system of equations:

$$3X + 2Y = 3 \times 3 + 2 \times 2 = 13$$
$$4Y - 2X = 4 \times 2 - 2 \times 3 = 2$$

(6) Verification completed and solutions found.

3. An Example

Suppose that you have $10 with which you can buy apples (A) and oranges (R). Also, assume that your bag can hold only 12 items – such as 12 apples, or 12 oranges, or some combination of apples and oranges. If the apple price is $1 and the orange price is $0.50, how many apples and oranges can you buy with your $10 and carry them home in your bag?

Answer:

The Substitution Method:

(1) identify relevant information:

Budget Condition: $A + 0.5R = 10$
Bag-Size Condition: $A + R = 12$

(2) convert the Bag-Size Condition as:

$$A = 12 - R$$

(3) substitute $A = 12 - R$ into the Budget Condition as:

$$(12 - R) + 0.5R = 10$$

$$-0.5R = -2 \quad \rightarrow \quad R=4$$

(4) substitute R=4 into (2) above and find:

$$A = 12 - 4 = 8$$

(5) verify the answer of A=8 and R=4 by plugging them into the above two conditions as:

Budget condition: $8 + 0.5(4) = 10$
Bag-Size condition: $8 + 4 = 12$

Because both conditions are met, the answer is A=8 and R=4.

The Elimination Method:

(1) identify relevant information:

Budget Condition: $A + 0.5R = 10$
Bag-Size Condition: $A + R = 12$

(2) subtract the bottom equation from the top:

$$-0.5R = -2 \quad \rightarrow \quad R=4$$

(3) plug this R=4 into either one of the two conditions above:

$$A + 0.5(4) = 10 \quad \rightarrow A=8$$

Or $A + (4) = 12$ → $A=8$

(4) verify the answer of A=8 and R=4 by plugging them into the above two conditions as:

Budget condition: $8 + 0.5(4) = 10$
Bag-Size condition: $8 + 4 = 12$

Because both conditions are met, the answer is A=8 and R=4.

4. Solve the following simultaneous equations by using both the substitution and elimination methods:

a. $20X + 4Y = 280$
$10Y – 9X = 110$

Answer: X=10 and Y=20

b. $2X + 7 = 5Y$
$3Y + 7 = 4X$

Answer: X=4 and Y=3

c. $(1/3)X – (1/4)Y = –37.5$
$3Y – 5X = 330$

Answer: X=120 and Y=310

Note that there is no way of telling which solution method – the substitution or the elimination – is superior to the other. Even though the elimination method is often preferred, it is the experience and preference of the solver that will decide which method would be used.

M. Examples of Algebra Problems

1. For your charity organization, you had served 300 customers who bought either one hot dog at \$1.50 or one hamburger at \$2.50, but never the two together. If your total sales of hot dogs (HD) and hamburgers (HB) were \$650 for the day, how many hot dogs and hamburgers did you sell?

2. You are offered an identical sales manager job. However, Company A offers you a base salary of \$30,000 plus a year-end bonus of 1% of the gross sales you make for that year. Company B, on the other hand, offers a base salary of \$24,000 plus a year-end bonus of 2% of the gross sales you make for that year.

a. Which company would you work for?

b. If you can achieve a total sale of $1,000,000 for either A or B, which company would you work for?

3. A fitness club offers two aerobics classes. In Class A, 30 people are currently attending and attendance is growing 3 people per month. In Class B, 20 people are regularly attending and growing at a rate of 5 people per month. Predict when the number of people in each class will be the same.

4. Everybody knows that Dr. Choi is the best instructor at DePaul. When a student in GSB 420 asked about the midterm exam, he said the following:

a. "The midterm exam will have a total of 100 points and contain 35 problems. Each problem is worth either 2 points or 5 points. Now, you have to figure out how many problems of each value there are in the midterm exam."

b. "The midterm exam will have a total of 108 points and there are twice as many 5-point problems than 2-point problems. Each problem is worth either 2 points or 5 points. Now, you have to figure out how many problems of each value there are in the midterm exam."

5. Your boss asked you to prepare a company party for 20 employees with a budget of $500. You have a choice of ordering a steak dinner at $30 per person or a chicken dinner at $25 per person. (All tips are included in the price of the meal.)

a. How many steak dinners and chicken dinners can you order for the party by using up the budget?

b. How many steak dinners and chicken dinners can you order for the party if the budget increases to $550?

6. Your father just received a notice from the Social Security Administration saying the following:

"If you retire at age 62, your monthly social security payment will be $1500. If you retire at age 66, your monthly social security payment will be $2100."

a. Your father is asking you to help decide which option to take. What would you tell him? Do not consider the time value of money. (Hint:

Calculate the age at which the social security income received will be the same.)

b. The Social Security Administration has given your father one more option as: “If you retire at age 70, your monthly social security payment will be $2800.” What would you now tell him? Do not consider the time value of money. (Hint: Calculate the age at which the social security income received will be the same.)

Answers to Above Examples of Algebra Problems

1. Quantity Condition: HD + HB = 300
Sales Condition: 1.50 HD + 2.50 HB = 650

Solving these two equations simultaneously, we find

HD* = 100 and HB* = 200

2.a. We have to identify the break-even sales (S) for both companies as follows:

Compensation from A = $30,000 + 0.01 S
Compensation from B = $24,000 + 0.02 S

Therefore,

Compensation from A = Compensation from B

$30,000 + 0.01 S = $24,000 + 0.02 S
S* = $600,000

Conclusion: If you think you can sell more than $600,000, you had better work for B. Otherwise, work for A.

2.b. Since you can sell more than $600,000, such as $1 million, work for B and possibly realize a total compensation of $44,000 (=$24,000 + 0.02 x ($1 million)). If you work for A, you would receive $40,000 (=$30,000 + 0.01 x ($1 million)).

3. Attendance in A = 30 + 3(Months)
Attendance in B = 20 + 5(Months)

Attendance in A = Attendance in B

Therefore, 30 + 3(Months) = 20 + 5(Months)

Months* = 5

4.a. Total Points: $2X + 5Y = 100$
Number of Problems: $X + Y = 35$

where X = the number of 2 point problems and Y = the number of 5 point problems.

Therefore, $X^*=25$ and $Y^*=10$

4.b. Total Points: $2X + 5Y = 108$
Number of Problems: $2X = Y$

where X = the number of 2 point problems and Y = the number of 5 point problems.

Therefore, $X^*=9$ and $Y^*=18$

5.a. Total Number of Employees: $S + C = 20$
Budget: $30S + 25C = 500$

where S = number of steak dinner and C = chicken dinner.

Therefore, $C^*=20$ and $S^*=0$

5.b. Total Number of Employees: $S + C = 20$
Budget: $30S + 25C = 550$

where S = number of steak dinner and C = chicken dinner.

Therefore, $C^*=10$ and $S^*=10$

6.a. Total Receipt between 62 and X = $(X – 62)*1500*12$
Total Receipt between 66 and X = $(X – 66)*2100*12$
Total Receipt between 62 and X = Total Receipt between 66 and X

That is, $(X – 62)*1500*12 = (X – 66)*2100*12$
Therefore, $X^* = 76$

That is, if your father can live longer than 76 of age, he should start receiving the social security payment at 66 of age. Otherwise, he should retire at 62.

6.b. If the retirement decision is between 62 vs. 70:

(X – 62)*2100*12 = (X – 70)*2800*12

Therefore, X* = 79.23

That is, if your father can live longer than 79.23 of age, he should retire at 70 of age. Otherwise, he should retire at 62.

If the retirement decision is between 66 vs. 70:

(X – 66)*2100*12 = (X – 70)*2800*12

Therefore, X* = 82

That is, if your father can live longer than 82 of age, he should retire at 70 of age. Otherwise, he should retire at 66.